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ON THE CONVEXITY OF THE HADLEY-WHITIN LOST SALES INVENTORY MODEL AND THE CHARACTERIZATION OF ITS MINIMUM SOLUTION.

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THESIS

ON THE CONVEXITY OF THE HADLEY-WHITIN
LOST SALES INVENTORY MODEL AND THE
CHARACTERIZATION OF ITS MINIMUM SOLUTION

by

Robert Gary Trapnell

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Thesis Advisor: P.W. Zehna

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completely characterized.

On the Convexity of the Hadley-Whitin Lost Sales Inventory Model and the Characterization of its Minimum Solution

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ABSTRACT

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TABLE OF CONTENTS

I. INTRODUCTION

The text by Hadley and Whitin [Ref. 1] is a standard reference on mathematical inventory models. A particular class of models presented therein which has received wide acceptance is known as the lot-size, reorder point, or (Q,r), model, where r is defined as that inventory level at which a reorder for a quantity Q is initiated. Both Q and r are assumed to be continuous variables. The authors begin their treatment of stochastic customer demand by deriving two models referred to as the backorders case and the lost sales case, respectively. In the backorders case, all demand occurring when the system is out of stock is eventually filled, whereas in the lost sales case, this demand is lost forever.

The models derived by Hadley and Whitin consist of long-run average annual variable cost functions, both labeled $K(Q,r)$. Each function consists of a sum of three cost expressions, the first representing a set-up or ordering cost, the second an inventory carrying cost, and the third a backorders or lost sales cost as appropriate. The authors assert that these functions are convex in the variables Q and r and their partial derivatives are equated to zero to determine unique and global minimum costs in each case. In particular, they claim these functions are strictly convex in the case where lead time demand (that demand occurring

between the reorder point and arrival of that order) is normally distributed. These assertions are unproven however, and are left to the reader as a series of exercises.

The first exercise asks the reader to show that a particular term which occurs in both models, say J(Q,r), is convex. It is obvious that the other terms are convex. It would then follow that K(Q,r) is convex in Q and r from additivity. However, it was pointed out in 1964 by Veinott [Ref. 2] that J(Q,r) is not always convex. In 1969, Brooks and Lu [Ref. 3] considered the convexity in the backorders model and showed, for normally distributed lead time demand, that J(Q,r) is convex for all $r \geq \mu$, the mean lead time demand. In 1975, the nonconvexity of the backorders cost function and the nonuniqueness of its solution was demonstrated by Minh [Ref. 4] for the case where lead time demand is normally distributed. In particular, Minh showed through counter-examples that K(Q,r) is not convex and, if a solution exists, its partial derivatives equated to zero have exactly two solutions vice one as claimed by Hadley and Whitin. Minh also succeeded in classifying one of the solutions as a relative minimum and the other as a saddle point. Further, Minh showed that feasible minimum solutions can be obtained where $r < \mu$.

As Minh has suggested, his results have created doubts as to Hadley and Whitin's claim that the lost sales cost function is strictly convex for the case of normally distributed lead time demand. Indeed, the only mathematical

difference between the two models consists of an additional term in the inventory carrying cost expression of the lost sales cost function. This term accounts for the additional cost incurred by carrying as inventory that portion of an arriving lot which would have been immediately dispatched to satisfy accumulated backorder demand, if such were permitted. This is consistent with the authors' assumption that the expected number of lost sales is equivalent to the expected number of backorders, all other factors being equal.

If, as will be shown in this thesis, K is not convex as claimed, then of course a different set of arguments must be provided for its minimization. In particular, the constraint imposed by assuming a nonnegative reorder point will also have to be examined. Hadley and Whitin ignore this constraint in their development, leaving the impression that the solution is always to be found in the region $r > 0$, $Q > 0$. Indeed, they are not clear as to their intent regarding r at all. In the beginning of their derivation, they require r to be positive; in the exercises dealing with convexity however, r is allowed to be nonnegative. Since r is assumed to be a continuous variable, there seems to be no rational basis for excluding $r = 0$ and it will be assumed herein that r is to be treated as a nonnegative variable.

It is the purpose of this thesis to clarify the issue of convexity for the lost sales case and completely characterize

its solution, including the effect of the constraint on r, assuming a normally distributed lead time demand. Section II will show that K is not convex and that the nonnegativity constraint on r precludes Hadley and Whitin's claim of existence in that the simultaneous solution to the partial derivations of K may be outside its domain of definition. Section III will present a series of lemmas and theorems in which the minimum solution is completely characterized. Section IV will briefly discuss these results.

For a normally distributed lead time demand, Hadley and Whitin define the lost sales cost function as follows:

$$K(Q,r) = \frac{\lambda A}{Q} + IC\left[\frac{Q}{2} + r - \mu\right] + [IC + \pi\lambda]\left[(\mu - r)\Phi\left(\frac{r-\mu}{\sigma}\right) + \sigma\phi\left(\frac{r-\mu}{\sigma}\right)\right] \quad .$$

The parameters are defined as follows:

- A is the set-up, or ordering cost.
- C is the unit cost.
- I is the inventory carrying cost per unit per year.
- π is the cost of a lost sale.
- λ is the mean annual demand.
- σ is the standard deviation of the normally distributed lead time random variable.
- μ is the mean of that random variable.
- $\phi(z)$ is the standard normal density function.
- $\Phi(z)$ is its complimentary cumulative distribution function, i.e. $\Phi(z) = \int_z^\infty \phi(x)dx$.

For ease in subsequent algebraic manipulation, $K(Q,r)$ will be rewritten as

$$K(z,Q) = \frac{\lambda A}{Q} + IC[\frac{Q}{2} + \sigma z + \eta(z)] + \frac{\pi\lambda\eta(z)}{Q}$$

where $z = \frac{r-\mu}{\sigma}$ and $\eta(z) = \sigma[\phi(z) - z\Phi(z)]$. In this notation, the set-up cost, inventory carrying cost, and lost sales costs are clearly identifiable. The domain of definition for K, as a function of z and Q, is defined by the restrictions $z \geq -\frac{\mu}{\sigma}$, $Q > 0$.

II. COUNTEREXAMPLES TO CONVEXITY AND EXISTENCE OF SOLUTION

A function of two or more variables is strictly convex if and only if its Hessian matrix is everywhere positive definite. The Hessian matrix is positive definite if and only if its proper diagonal terms and determinant are strictly positive. The Hessian for K(z,Q) is derived below.

The Hessian matrix is defined as

$$\Omega = \begin{bmatrix} \dfrac{\partial^2 K}{\partial z^2} & \dfrac{\partial^2 K}{\partial z \partial Q} \\ & \\ \dfrac{\partial^2 K}{\partial Q \partial z} & \dfrac{\partial^2 K}{\partial Q^2} \end{bmatrix}$$

Since K(z,Q) is continuously differentiable over the region $z > -\frac{u}{\sigma}$, $Q > 0$,

$$\frac{\partial^2 K}{\partial z \partial Q} = \frac{\partial^2 K}{\partial Q \partial z} .$$

For subsequent derivations, K will be written as

$$K(z,Q) = \frac{\lambda}{Q}[A + \pi\eta(z)] + IC[\frac{Q}{2} + \sigma z + \eta(z)].$$

Then,

$$\frac{\partial K}{\partial z} = \frac{\pi\lambda}{Q}\eta'(z) + IC[\sigma + \eta'(z)]$$

where

$$\eta'(z) = -\sigma\Phi(z) .$$

Thus,

$$\frac{\partial k}{\partial z} = -\frac{\pi\lambda\sigma\Phi(z)}{Q} + IC\sigma[1 - \Phi(z)].$$

Hence,

$$\frac{\partial K}{\partial z} = 0 \quad \text{implies} \quad \Phi(z^*) = \frac{Q^*IC}{Q^*IC + \pi\lambda} \qquad (1)$$

$$\frac{\partial^2 K}{\partial z^2} = \frac{\pi\lambda\sigma\phi(z)}{Q} + IC\sigma\phi(z)$$

$$= \frac{\sigma\phi(z)}{Q}[QIC + \pi\lambda]$$

$$\frac{\partial^2 K}{\partial Q\partial z} = \frac{\pi\lambda\sigma\Phi(z)}{Q^2} = \frac{\partial^2 K}{\partial z\partial Q}$$

$$\frac{\partial K}{\partial Q} = -\frac{\lambda}{Q^2}[A + \pi\eta(z)] + \frac{IC}{2} .$$

Then,

$$\frac{\partial K}{\partial Q} = 0 \quad \text{implies} \quad Q^* = \sqrt{\frac{2\lambda[A + \pi\eta(z^*)]}{IC}} \qquad (2)$$

$$\frac{\partial^2 K}{\partial Q^2} = \frac{2\lambda}{Q^3}[A + \pi\eta(z)]$$

The Hessian matrix is therefore:

$$\Omega = \begin{bmatrix} \dfrac{\sigma\phi(z)\,[QIC + \pi\lambda]}{Q} & \dfrac{\pi\lambda\sigma\Phi(z)}{Q^2} \\[2ex] \dfrac{\pi\lambda\sigma\Phi(z)}{Q^2} & \dfrac{2\lambda\,[A + \pi\eta(z)]}{Q^3} \end{bmatrix}$$

It is shown in the Appendix that the function $\eta(z)$ is strictly positive. Thus the proper diagonal terms of Ω are strictly positive and Ω is positive definite if and only if det Ω (determinant of Ω) is positive. The determinant of Ω is given by

$$\det \Omega = \frac{\lambda\sigma}{Q^4}\left[2\phi(z)\,[A + \pi\eta(z)]\,[QIC + \pi\lambda] - \pi^2\lambda\sigma\Phi^2(z)\right]$$

Define the expression contained within the outer brackets as $V(z,Q)$. Since $\frac{\lambda\sigma}{Q^4} > 0$, det $\Omega > 0$ if and only if

$$V(z,Q) = 2\phi(z)\,[A + \pi\eta(z)]\,[QIC + \pi\lambda] - \pi^2\lambda\sigma\Phi^2(z) > 0.$$

With this result, a counterexample to the claim that $K(z,Q)$ is strictly convex is available. The parameters utilized are from an example given by Hadley and Whitin where the lead time demand is normally distributed and are specified as $A = 4000$, $C = 50$, $I = 0.20$, $\lambda = 1600$, $\pi = 2000$, $\sigma = 50$, $\mu = 750$. Then, for the policy $r = 700$ $(z = -1.0)$ and $Q = 10$, $\phi(z) \doteq 0.2420$, $\Phi(z) \doteq 0.8413$ and so,

$$V(z,Q) \doteq 2(.2420)\Big[[4000 + 2000(50)(.2420 + .8413)] \times [(10)(.20)(50) + (2000)(1600)] \Big] - (2000)^2(1600)(50)(.8413)^2$$

$$\doteq 1.7398214 \times 10^{11} - 2.2649142 \times 10^{11} < 0.$$

Thus, $K(z,Q)$ is not strictly convex throughout the region $r > 0$, $Q > 0$ as claimed.

As for a counterexample to the existence of the solution, choose the following parameters: $A = 25{,}000$, $C = 10{,}000$, $I = 0.99$, $\pi = 10$, $\lambda = 5$, $\sigma = 10$, and $\mu = 30$. For these parameters, the iteration procedure described by Hadley and Whitin produces a solution $(z^*,Q^*) \doteq (-3.10, 5.06)$. It may be verified by results to be subsequently derived that $V(z^*,Q^*)$ is positive. It will be observed, however, that the reorder point r corresponding to $z^* = -3.10$ is $r = z^*\sigma + \mu = -1.0$ which is outside the domain of definition for $K(Q,r)$ as defined by Hadley and Whitin. Basically, the counterexample is a consequence of the fact that the solution (z^*,Q^*) is independent of μ, the mean lead time demand.

III. CHARACTERIZATION OF THE MINIMUM SOLUTION; PART A

It is convenient to separate out as lemmas certain isolated results that support the main results presented in this thesis. The first four of these lemmas are stated in this section and derived in the Appendix. They lead to an intermediate result stated as THEOREM 1, a theorem that demonstrates the uniqueness of the solution (z^*,Q^*) to equations (1) and (2) and establishes that solution to be an unconstrained relative minimum for K.

LEMMAs 5 thru 7, also stated in this section and derived in the Appendix, lead to THEOREM 2, a theorem establishing that, for all points (z,Q) where $z \neq z^*$, $K(z^*,Q^*) < K(z,Q)$. The main result will then be presented as THEOREM 3 in which the minimum solution is completely characterized.

Let a set of parameters A, C, I, π, λ, σ, and μ, all greater than zero, be given. Define the following functions for $-\infty < z < \infty$:

$$A(z) = \frac{\pi^2 \lambda \Phi^2(z)}{2IC[1 - \Phi(z)]^2} - \pi\eta(z)$$

$$G(z) = 3\phi(z) + z - z\Phi(z)$$

$$H(z) = \frac{\phi(z)}{[1 - \Phi(z)]^3}$$

LEMMA 1: Let (z^*, Q^*) be a solution to equations (1) and (2). Then,

(a) $A = A(z^*)$

(b) $V(z^*, Q^*) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0$ if and only if

$$H(z^*) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} \frac{IC\sigma}{\pi\lambda}$$

LEMMA 2: (a) The function $G(z)$ is strictly positive.

(b) The function $H(z)$ is strictly monotone decreasing.

(c) The equation $H(z) = \frac{IC\sigma}{\pi\lambda}$ has one and only one solution.

LEMMA 3: The function $A(z)$ has a unique minimum at z_o where z_o is the unique solution to the equation $H(z) = \frac{IC\sigma}{\pi\lambda}$.

LEMMA 4: (a) $\lim_{z \to \infty} A(z) = 0$

(b) $\lim_{z \to -\infty} A(z) = \infty$

(c) $A(z_o) < 0$

(d) There exists one and only one $z_1 < z_o$ such that $A(z_1) = 0$. Moreover, if $z > z_1$, $A(z) < 0$.

THEOREM 1: There exists one and only one (z^*,Q^*) which is a simultaneous solution to equations (1) and (2) and that solution determines a relative minimum for K.

PROOF: From LEMMA 1, let z^* be chosen so that $A(z^*) = A > 0$. Then $z^* < z_1$ from LEMMA 4(d). From the strict monotonicity of $A(z)$ for $-\infty < z \leq z_1$, it then follows that z^* is unique. Let Q^* then be selected according to either of equations (1) or (2). Then, from LEMMA 1, (z^*,Q^*) is a unique solution to these equations. To show that $K(z^*,Q^*)$ is a relative minimum, note that $z^* < z_1 < z_o$. Then, by LEMMA 3, $H(z^*) > \frac{IC\sigma}{\pi\lambda}$. But LEMMA 1(b) establishes this as a necessary and sufficient condition for positive definiteness of the Hessian of K at (z^*,Q^*). This positive definiteness establishes that $K(z^*,Q^*)$ is a relative minimum of K and uniqueness has already been established.

Q.E.D.

III. CHARACTERIZATION OF THE MINIMUM SOLUTION; PART B

The results of the previous lemmas are depicted graphically in Figure 1. The functions A(z) and H(z) and the lines A and $\frac{IC\sigma}{\pi\lambda}$ are shown for perspective in that they represent no particular set of parameters. It should be noted that z*, here shown negative, may be positive or negative.

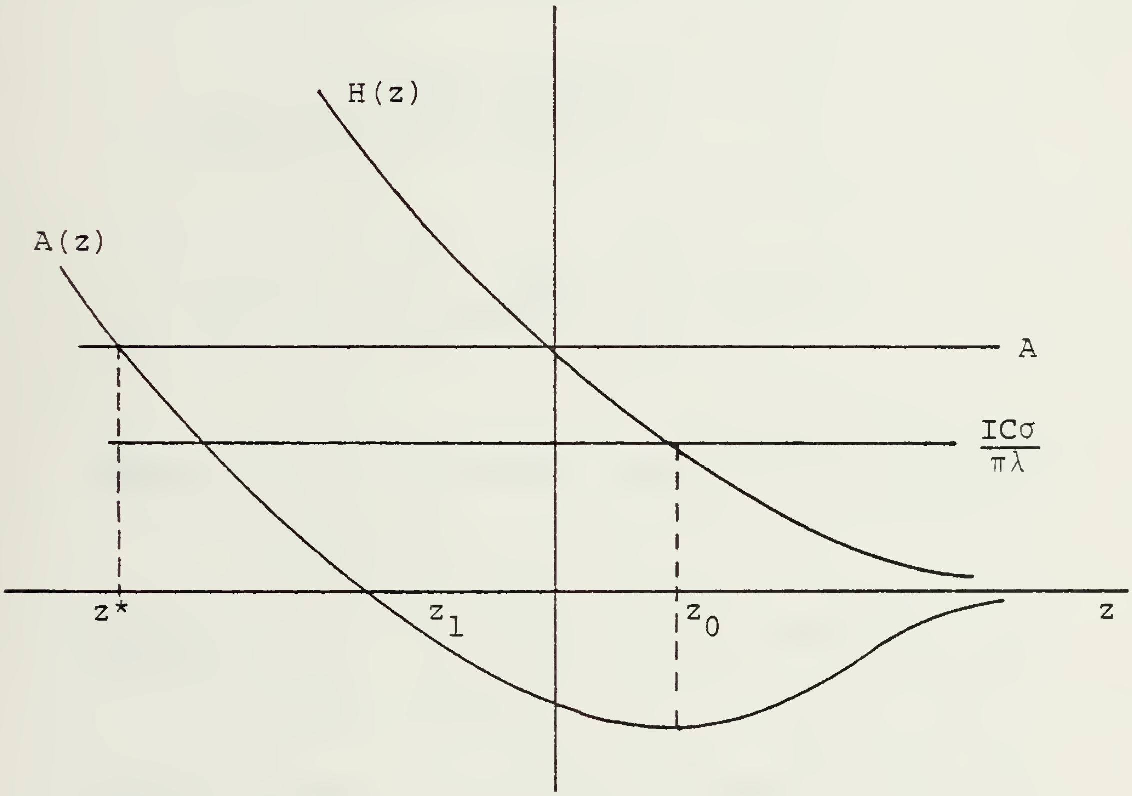

Figure 1

It has been established in PART A that K(z*,Q*) constitutes a relative minimum and that (z*,Q*) is unique. In this part it will be demonstrated that K(z*,Q*) does

constitute an unconstrained global minimum. However, it was shown in Section II that the constraint $r \geq 0$ ($z \geq -\frac{\mu}{\sigma}$) may be binding. Therefore that constraint must be considered in the minimization of K.

For the succeeding lemmas and theorems, it will be convenient to define the following functions for $Q > 0$, $-\infty < z < \infty$.

$$Q_1(z) = \frac{\pi\lambda\Phi(z)}{IC[1 - \Phi(z)]}$$

$$Q_2(z) = \sqrt{\frac{2\lambda[A + \pi\eta(z)]}{IC}}$$

$$z_1(Q) = \Phi^{-1}\left[\frac{QIC}{QIC + \pi\lambda}\right] = Q_1^{-1}(Q)$$

<u>LEMMA 5</u>: Let z be fixed. Then, $K(z,Q_2(z)) \leq K(z,Q)$ for all Q.

<u>LEMMA 6</u>: If $z < z^*$, $Q_2(z) < Q_1(z)$, and if $z > z^*$, $Q_2(z) > Q_1(z)$.

<u>LEMMA 7</u>: Let $Q > 0$ be fixed. Then, if $z < z_1(Q)$, $\frac{\partial K}{\partial z} < 0$ and, if $z > z_1(Q)$, $\frac{\partial K}{\partial z} > 0$.

<u>THEOREM 2</u>: If $z \neq z^*$, then $K(z^*,Q^*) < K(z,Q)$ for all z and $Q > 0$.

PROOF: Let z be fixed and $v = (\cos\theta, \sin\theta)$ be a unit vector from $(z, Q_2(z))$ in the direction of (z^*, Q^*). The directional derivative of K at $(z, Q_2(z))$ may be written as

$$D_K\, v = \frac{\partial K}{\partial z}\cos\theta + \frac{\partial K}{\partial Q}\sin\theta$$

$$= \frac{\partial K}{\partial z}\cos\theta$$

since $\frac{\partial K}{\partial Q} = 0$ at $(z, Q_2(z))$. The proof now proceeds by cases.

Case 1: $z < z^*$. Since $Q_2(z)$ is strictly monotone decreasing, $Q_2(z) > Q_2(z^*)$, and thus $\frac{3\pi}{2} < \theta < 2\pi$. Hence, $\cos\theta > 0$. Let $Q_1 = Q_1(z)$ and $Q_2 = Q_2(z)$. By LEMMA 8, $Q_2 < Q_1$, in which case $z_1(Q_2) > z_1(Q_1)$ since z_1 is strictly monotone decreasing. Since $z_1(Q_1) = z$, $z_1(Q_2) > z$ which implies $\frac{\partial K}{\partial z} < 0$ by LEMMA 9. Hence, $D_K v < 0$ for $z < z^*$.

Case 2: $z > z^*$. Since $Q_2(z)$ is strictly monotone decreasing, $Q_2(z) < Q_2(z^*)$, and thus $\frac{\pi}{2} < \theta < \pi$. Hence, $\cos\theta < 0$. Let $Q_1 = Q_1(z)$ and $Q_2 = Q_2(z)$. By LEMMA 8, $Q_2 > Q_1$, in which case $z_1(Q_2) < z_1(Q_1)$ since z_1 is strictly monotone decreasing. Since $z_1(Q_1) = z$, $z_1(Q_2) < z$ which implies $\frac{\partial K}{\partial z} > 0$ by LEMMA 9. Hence, $D_K v < 0$ for $z > z^*$.

In all cases, $D_K v < 0$ so that K is strictly decreasing from any point $(z, Q_2(z))$ toward (z^*, Q^*). Thus $K(z^*, Q^*) < K(z, Q_2(z))$ and, by LEMMA 5, $K(z, Q_2(z)) \leq K(z, Q)$ for any $Q > 0$.

Q.E.D.

<u>THEOREM 3</u>: If $z^* \geq -\frac{\mu}{\sigma}$, then $K(z^*, Q^*)$ is the global minimum value of K. If $z^* < -\frac{\mu}{\sigma}$, then $K(-\frac{\mu}{\sigma}, Q_2(-\mu/\sigma))$ is the global minimum value of K.

PROOF: If $z^* \geq -\frac{\mu}{\sigma}$, then the corresponding value of r is nonnegative and r is in the domain of definition of K. From THEOREM 2, $K(z^*, Q^*) < K(z, Q)$ for all $z \neq z^*$ and $Q > 0$. Thus, $K(z^*, Q^*)$ is a global minimum if $Q > 0$ and $z^* \geq -\frac{\mu}{\sigma}$.

If $z^* < -\frac{\mu}{\sigma}$, then the corresponding value of r is negative and is thus outside the domain of definition of K. Hence, $K(z^*, Q^*)$ cannot be optimal. However, LEMMA 5 has established that, for any fixed z, $K(z, Q_2(z)) \leq K(z, Q)$ and, by THEOREM 2, K is strictly decreasing from any point $(z, Q_2(z))$ toward (z^*, Q^*). Hence, $K(-\frac{\mu}{\sigma}, Q_2(-\frac{\mu}{\sigma})) \leq K(z, Q_2(z))$ for any $z \geq -\frac{\mu}{\sigma}$ $(r \geq 0)$. Thus, the minimum value of K occurs on the boundary $z = -\frac{\mu}{\sigma}$ and, for $z^* < -\frac{\mu}{\sigma}$, K is minimized at $K(-\frac{\mu}{\sigma}, Q_2(-\frac{\mu}{\sigma}))$.

Q.E.D.

IV. DISCUSSION

The results of this thesis have resolved a long-standing question concerning the solution to the lost sales case of the Hadley-Whitin (Q,r) model. Since the model is widely used by practitioners, it is an issue that needs resolution, for the entire argument given by the authors is based on the supposed convexity of the objective function. That not being the case, it is necessary to supply a different argument for the solution. Convexity aside, the boundary constraint can be binding and appears to have been completely ignored by the authors in their development. This thesis has clarified both these issues.

Beyond the mere clarification of the results, this thesis exemplifies the fact that attention to the underlying mathematical detail and rigor cannot be ignored in modeling a problem. Fortunately, in this case, the solution remains partially valid but for a different set of reasons. This could only have been discovered by a close and detailed examination of the mathematics of the model. In the process of that examination, the role of the boundary condition has revealed itself and allows for complete characterization of the solution.

As for recommendations for further study, it should be pointed out that only the case of normally distributed lead time demand has been considered. There are, however, many

other assumptions that might be made concerning this demand. Among those of special interest to Navy applications are those of the Poisson and negative binomial distributions. An investigation similar to that presented here for at least these two cases would seem to be called for and is a recommendation of this thesis.

APPENDIX A

PROOFS OF LEMMAS

This Appendix contains the proofs of LEMMAs 1 thru 7 inclusive. Prior to their proofs, it will be convenient to prove four auxiliary lemmas which are appealed to in the main argument. For clarity, the auxiliary lemmas will be numbered with the letter "A" following the number.

LEMMA 1A: The function $\eta(z)$ is strictly monotone decreasing for $-\infty < z < \infty$ and everywhere strictly positive over that region.

PROOF: $\eta(z) = \sigma u(z)$

where

$$u(z) = \phi(z) - z\Phi(z).$$

Now,

$$u'(z) = -z\phi(z) - z(-\phi(z)) - \Phi(z)$$

$$= -\Phi(z) < 0 \qquad \text{for all } z.$$

$$u''(z) = \phi(z) > 0 \qquad \text{for all } z.$$

Clearly,

$$\lim_{z\to-\infty} u(z) = \lim_{z\to-\infty} (\phi(z)-z\Phi(z)) = 0 + \infty = +\infty.$$

And

$$\lim_{z\to\infty} u(z) = \lim_{z\to\infty} (\phi(z)-z\Phi(z)) = -\lim_{z\to\infty} z\Phi(z)$$

$$= - \lim_{z\to\infty} \frac{z}{\frac{1}{\Phi(z)}}$$

Since this limit results in the indeterminant form $\frac{\infty}{\infty}$, L'Hôpital's Rule is applied repeatedly.

$$- \lim_{z\to\infty} \frac{z}{\frac{1}{\Phi(z)}} = - \lim_{z\to\infty} \frac{1}{\frac{\phi(z)}{\Phi^2(z)}} = - \lim_{z\to\infty} \frac{\Phi^2(z)}{\phi(z)}$$

$$= - \lim_{z\to\infty} \frac{-2\phi(z)\Phi(z)}{-z\phi(z)}$$

$$= - \lim_{z\to\infty} \frac{2\Phi(z)}{z} = 0.$$

Thus $\eta(z) = \sigma u(z)$ is a strictly convex decreasing function approaching 0 as $z\to+\infty$ and hence $\eta(z) > 0$ for all z.

LEMMA 2A: $Q_1(z) = \frac{\pi\lambda\Phi(z)}{IC[1 - \Phi(z)]}$ is strictly monotone decreasing and everywhere strictly positive.

PROOF: $$Q_1'(z) = \frac{\pi\lambda}{IC}\left[\frac{-\Phi(z)\phi(z)}{[1 - \Phi(z)]^2} + \frac{-\phi(z)}{[1 - \Phi(z)]}\right]$$

$$= \frac{-\pi\lambda\phi(z)}{IC[1 - \Phi(z)]^2} < 0 \quad \text{for all } z.$$

$Q_1(z)$ is everywhere strictly positive by inspection.

Q.E.D.

LEMMA 3A: $Q_2(z) = \sqrt{\frac{2\lambda[A + \pi\eta(z)]}{IC}}$ is strictly monotone decreasing and everywhere strictly positive.

PROOF: Squaring $Q_2(z)$ and differentiating implicitly,

$$Q_2'(z) = -\frac{\lambda\pi\sigma\Phi(z)}{Q_2(z)IC} < 0 \quad \text{for all } z$$

$Q_2(z)$ is everywhere strictly positive by inspection.

Q.E.D.

LEMMA 4A: $z_1(Q) = \Phi^{-1}\left[\frac{QIC}{QIC + \pi\lambda}\right]$ is strictly monotone decreasing.

PROOF: $$Q_1(z) = \frac{\pi\lambda\Phi(z)}{IC[1 - \Phi(z)]}$$

Thus,

$$\Phi(z) = \frac{QIC}{QIC + \pi\lambda}$$

or,

$$z = \Phi^{-1}\left[\frac{QIC}{QIC + \pi\lambda}\right]$$

$$= z_1(Q) = Q_1^{-1}(Q)$$

Since $z_1(Q)$ is the functional inverse of $Q_1(z)$ and $Q_1(z)$ is strictly monotone decreasing, $z_1(Q)$ is strictly monotone decreasing.

Q.E.D.

<u>LEMMA 1</u>: (a) $A = A(z^*)$.

(b) $V(z^*,Q^*) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0$ if and only if

$$H(z^*) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} \frac{IC\sigma}{\pi\lambda} .$$

Proof (a). Equating the two expressions for Q^* given by equations (1) and (2) yields

$$\frac{\pi\lambda\Phi(z^*)}{IC[1 - \Phi(z^*)]} = \sqrt{\frac{2\lambda[A + \pi\eta(z^*)]}{IC}}$$

or,

$$\frac{\pi^2\lambda\Phi^2(z^*)}{IC[1 - \Phi(z^*)]^2} = 2[A + \pi\eta(z^*)]$$

Solving for A,

$$\frac{\pi^2\lambda\Phi^2(z^*)}{2IC[1 - \Phi(z^*)]^2} - \pi\eta(z^*) = A$$

that is,

$$A(z^*) = A$$

Proof (b). $V(z^*,Q^*) = 2\phi(z^*)[A+\pi\eta(z^*)][Q^*IC+\pi\lambda] - \pi^2\lambda\sigma\Phi^2(z^*)$

Substituting $A(z^*)$ for A from part (a) and

$Q^* = \dfrac{\pi\lambda\Phi(z^*)}{IC[1 - \Phi(z^*)]}$ from equation 1,

$$V(z^*,Q^*) = 2\phi(z^*)\left[\frac{\pi^2\lambda\Phi^2(z^*)}{2IC[1 - \Phi(z^*)]^2}\right]\left[\frac{\pi\lambda\Phi(z^*)}{[1 - \Phi(z^*)]} + \pi\lambda\right]$$

$$- \pi^2\lambda\sigma\Phi^2(z^*)$$

$$= 2\phi(z^*)\left[\frac{\pi^2\lambda\Phi^2(z^*)}{2IC[1 - \Phi(z^*)]^2}\right]\left[\frac{\pi\lambda}{[1 - \Phi(z^*)]}\right]$$

$$- \pi^2\lambda\sigma\Phi^2(z^*)$$

$$V(z^*,Q^*) = \pi^2\lambda\Phi^2(z^*) \quad \frac{\pi\lambda\Phi(z^*)}{IC[1 - \Phi(z^*)]^3} - \sigma$$

Since $\pi^2\lambda\Phi^2(z^*) > 0$, $V(z^*,Q^*) \left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\} 0$ if and only if

$$\frac{\pi\lambda\phi(z^*)}{IC[1 - \Phi(z)]^3} - \sigma \left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\} 0.$$

Since $\frac{IC}{\pi\lambda} > 0$, this condition may be written as

$$\frac{\phi(z^*)}{[1 - \Phi(z^*)]^3} \equiv H(z^*) \left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\} \frac{IC\sigma}{\pi\lambda}$$

Q.E.D.

<u>LEMMA 2</u>: (a) The function $G(z) = 3\phi(z) + z - z\Phi(z)$ is strictly positive.

(b) The function $H(z) = \dfrac{\phi(z)}{[1 - \Phi(z)]^3}$ is strictly monotone decreasing.

(c) The equation $H(z) = \frac{IC\sigma}{\pi\lambda}$ has one and only one solution.

Proof (a). The proof proceeds by cases.

Case 1. $z \geq 0$.

$$G(z) = 3\phi(z) + z - z\Phi(z) = u(z) + 2\phi(z) + z.$$

Since $u(z) = \phi(z) - z\Phi(z)$ is strictly positive for $-\infty < z < \infty$, $G(z) > 0$.

Case 2. $z < 0$. Let $z = -\bar{z}$ where $\bar{z} > 0$. Then,

$$
\begin{aligned}
G(z) = G(-\bar{z}) &= 3\phi(-\bar{z}) - \bar{z} + \bar{z}\Phi(-\bar{z}) \\
&= 3\phi(\bar{z}) - \bar{z} + \bar{z}[1 - \Phi(\bar{z})] \\
&= 3\phi(\bar{z}) - \bar{z}\Phi(\bar{z}) \\
&= 2\phi(\bar{z}) + u(\bar{z}) > 0.
\end{aligned}
$$

Proof (b). Differentiating $H(z)$,

$$
\begin{aligned}
H'(z) &= \frac{-3\ \phi^2(z)}{[1 - \Phi(z)]^4} - \frac{z\phi(z)}{[1 - \Phi(z)]^3} \\
&= \frac{-\phi(z)}{[1 - \Phi(z)]^3} \quad \frac{3\phi(z)}{[1 - \Phi(z)]} + z \\
&= \frac{-\phi(z)}{[1 - \Phi(z)]^4} \quad 3\phi(z) + z - z\Phi(z)
\end{aligned}
$$

The expression in brackets is $G(z)$ which has been shown in part (a) to be strictly positive. Hence, $H'(z) < 0$ and $H(z)$ is strictly monotone decreasing.

Proof (c). Since $\lim\limits_{z\to\infty} \Phi(z) = 0$,

$$\lim_{z\to\infty} H(z) = \lim_{z\to\infty} \frac{\phi(z)}{[1 - \Phi(z)]^3} = 0$$

$$\lim_{z\to-\infty} H(z) = \lim_{z\to\infty} H(-z) = \lim_{z\to\infty} \frac{\phi(z)}{\Phi^3(z)}$$

This leads to the indeterminate form $\frac{0}{0}$. Applying L'Hôpital's Rule,

$$\lim_{z\to\infty} \frac{\phi(z)}{\Phi^3(z)} = \lim_{z\to\infty} \frac{-z\phi(z)}{-3\Phi^2(z)\phi(z)}$$

$$= \lim_{z\to\infty} \frac{z}{3\Phi^2(z)}$$

$$= \infty$$

Since $H(z)$ is everywhere positive and strictly monotone and $\frac{IC\sigma}{\pi\lambda} > 0$, the equation $H(z) = \frac{IC\sigma}{\pi\lambda}$ can have one and only one solution.

Q.E.D.

<u>LEMMA 3:</u> The function $A(z)$ has a global minimum at z_o where z_o is the solution to the equation $H(z) = \frac{IC\sigma}{\pi\lambda}$.

Proof: It is convenient to define the following:

$$g(z) = \frac{\Phi(z)}{[1 - \Phi(z)]}, \qquad a = \frac{\pi\lambda}{IC}.$$

A(z) can then be written as

$$A(z) = \pi\left[\frac{a}{2}\, g^2(z) - \eta(z)\right]$$

Then,

$$A'(z) = \pi\left[a\ g(z)\ g'(z) + \sigma\Phi(z)\right] .$$

Since $g'(z) = \dfrac{-\phi(z)}{[1 - \Phi(z)]^2}$,

$$A'(z) = \pi\left[\frac{-\pi\lambda\Phi(z)\phi(z)}{IC[1 - \Phi(z)]^3} + \sigma\Phi(z)\right]$$

$$= \frac{\pi^2\lambda\Phi(z)}{IC}\left[\frac{IC\sigma}{\pi\lambda} - \frac{\phi(z)}{[1 - \Phi(z)]^3}\right] .$$

Since $\dfrac{\pi^2\lambda\Phi(z)}{IC} > 0$, $A'(z) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0$ precisely when

$$H(z) \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} \frac{IC\sigma}{\pi\lambda} .$$

By LEMMA 2, there exists a unique z_o such that $H(z_o) = \dfrac{IC\sigma}{\pi\lambda}$. Moreover, from the strict monotonicity of H(z),

$$H(z) \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} \frac{IC\sigma}{\pi\lambda} \quad \text{for} \quad z \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} z_o.$$

It then follows that

$$A'(z) \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 0 \quad \text{for} \quad z \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} z_o \quad \text{which}$$

establishes z_o, the only critical value, as yielding a unique minimum for $A(z)$.

Q.E.D.

<u>LEMMA 4</u>: (a) $\lim_{z\to\infty} A(z) = 0$

(b) $\lim_{z\to-\infty} A(z) = \infty$

(c) $A(z_o) < 0$

(d) There exists one and only one $z_1 < z_o$ such that $A(z_1) = 0$. Moreover, if $z > z_1$, $A(z) < 0$.

Proof (a). $A(z) = \dfrac{\pi^2 \lambda \Phi^2(z)}{2IC[1 - \Phi(z)]^2} - \pi\sigma u(z)$

It has already been shown that $\lim_{z\to\infty} u(z) = 0$. Hence,

$$\lim_{z\to\infty} A(z) = \lim_{z\to\infty} \frac{\pi^2\lambda\Phi^2(z)}{2IC[1 - \Phi(z)]^2} - \lim_{z\to\infty} \pi\sigma u(z)$$

$$= 0.$$

Proof (b). $A(-z) = \frac{\pi\sigma}{a}\left[\frac{\Phi^2(-z)}{[1 - \Phi(-z)]^2} - a\,u(-z)\right]$

where

$$a = \frac{2IC\sigma}{\pi\lambda}$$

$$A(-z) = \frac{\pi\sigma}{a}\left[\frac{[1 - \Phi(z)]^2}{\Phi^2(z)} - a[\phi(-z) + z\Phi(-z)]\right]$$

$$= \frac{\pi\sigma}{a}\left[\frac{[1 - \Phi(z)]^2}{\Phi^2(z)} - a[u(z) + z]\right]$$

$$= \frac{\pi\sigma}{a}\left[\frac{[1 - \Phi(z)]^2}{\Phi^2(z)} - az\right] - \pi\sigma u(z)$$

Since $\lim\limits_{z\to-\infty} A(z) = \lim\limits_{z\to\infty} A(-z)$ and $\lim\limits_{z\to\infty} u(z) = 0$,

$$\lim_{z\to-\infty} A(z) = \lim_{z\to\infty} \frac{\pi\sigma}{a}\left[\frac{[1 - \Phi(z)]^2}{\Phi^2(z)} - az\right]$$

$$= \frac{\pi\sigma}{a} \lim_{z\to\infty} \frac{[1 - \Phi(z)]^2 - az\Phi^2(z)}{\Phi^2(z)} .$$

But, $\lim\limits_{z\to\infty} z\Phi^2(z) = \lim\limits_{z\to\infty} \frac{z}{\frac{1}{\Phi^2(z)}} = \lim\limits_{z\to\infty} \frac{1}{\frac{2\phi(z)}{\Phi^3(z)}}$

$$= \frac{1}{2} \lim_{z\to\infty} \frac{\Phi^3(z)}{\phi(z)} = \frac{1}{2} \lim_{z\to\infty} \frac{-3\Phi^2(z)\phi(z)}{-z\phi(z)}$$

$$= \frac{3}{2} \lim_{z\to\infty} \frac{\Phi^2(z)}{z} = 0$$

and $\lim_{z \to \infty} [1 - \Phi(z)]^2 = 1$. Hence,

$$\lim_{z \to -\infty} A(z) = \lim_{z \to \infty} A(-z) = \infty.$$

Proof (c). The proof is by contradiction. Assume $A(z_o) \geq 0$. Since $A(z)$ is strictly increasing on the interval $z > z_o$, $A(z) > A(z_o) \geq 0$ for all $z > z_o$. This implies $\lim_{z \to \infty} A(z) > 0$ contradicting the fact that $\lim_{z \to \infty} A(z) = 0$ established in part (a).

Proof (d). It has been shown in LEMMA 3 that $A'(z) < 0$ for $-\infty < z < z_o$ and in part (b) above that $\lim_{z \to -\infty} A(z) = \infty$. Since $A(z_o) < 0$, there must exist $-\infty < z_1 < z_o$ such that $A(z_1) = 0$. That z_1 is unique in the interval $-\infty < z \leq z_o$ follows from the strict monotonicity of $A(z)$ in this interval.

Q.E.D.

<u>LEMMA 5</u>: Let z be fixed. Then, $K(z, Q_2(z)) \leq K(z,Q)$ for all Q.

Proof: For any fixed z,

$$\frac{\partial K}{\partial Q} = \frac{-\lambda[A + \pi\eta(z)]}{Q^2} + \frac{IC}{2}.$$

$$\frac{\partial K}{\partial Q} = 0 \quad \text{implies} \quad Q = \sqrt{\frac{2\lambda[A + \pi\lambda(z)]}{IC}} = Q_2(z).$$

$$\frac{\partial^2 K}{\partial Q^2} = \frac{2\lambda[A + \pi\eta(z)]}{Q^3} > 0.$$

Thus, $K(z,\cdot)$ is a convex function with a unique minimum at $Q_2(z)$.

Q.E.D.

<u>LEMMA 6</u>: If $z < z^*$, $Q_2(z) < Q_1(z)$ and if $z > z^*$, $Q_2(z) > Q_1(z)$.

Proof: Define the following functions for $-\infty < z < \infty$:

$$D(z) = W_2(z) - W_1(z)$$

where

$$W_1(z) = Q_1^2(z) = \left[\frac{\pi\lambda}{IC}\right]^2 \frac{\Phi^2(z)}{[1 - \Phi(z)]^2}$$

and

$$W_2(z) = Q_2^2(z) = \frac{2\lambda[A + \pi\eta(z)]}{IC}$$

Then,

$$D'(z) = W_2'(z) - W_1'(z)$$

$$= \frac{-2\pi\lambda\sigma\Phi(z)}{IC} + 2\left[\frac{\pi\lambda}{IC}\right]^2 \Phi(z)\ H(z)$$

$$= 2\left[\frac{\pi\lambda}{IC}\right]^2 \Phi(z)\left[H(z) - \frac{IC\sigma}{\pi\lambda}\right]$$

Thus, by LEMMA 2,

$$D'(z) \quad \left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\} \quad 0 \quad \text{if and only if}$$

$$z \quad \left\{\begin{matrix} < \\ = \\ > \end{matrix}\right\} \quad z_o .$$

Hence, z_o is a relative maximum for D. Observe that, since $Q_1(z) > 0$ and $Q_2(z) > 0$,

$$Q_2(z) \quad \left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\} \quad Q_1(z) \quad \text{iff} \quad W_2(z) \quad \left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\} \quad W_1(z)$$

$$\text{iff} \quad D(z) \quad \left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\} \quad 0$$ and $D\ (z) = 0$ if and only if $z = z^*$.

Suppose $z < z^*$. Since $z^* < z_o$ and D is increasing on $(-\infty, z_o)$, $D(z) < D(z^*) = 0$. Hence, $Q_2(z) < Q_1(z)$.

Suppose $z > z^*$. If $z^* < z \leq z_o$, then $0 = D(z^*) < D(z)$ or $Q_2(z) > Q_1(z)$. If $z_o < z$, then, since D is decreasing on (z_o, ∞), and $\lim_{z \to \infty} D(z) = \frac{2\lambda A}{IC} > 0$, $D(z) > 0$ or $Q_2(z) > Q_1(z)$.

Q.E.D.

<u>LEMMA 7</u>: Let $Q > 0$ be fixed. If $z < z_1(Q)$, then $\frac{\partial K}{\partial z} < 0$, while if $z > z_1(Q)$, $\frac{\partial K}{\partial z} > 0$.

Proof: For any fixed Q,

$$\frac{\partial K}{\partial z} = IC\sigma - [IC + \frac{\pi\lambda}{Q}]\sigma\Phi(z)$$

$$\frac{\partial K}{\partial z} = 0 \quad \text{implies} \quad z = \Phi^{-1}\left[\frac{QIC}{QIC + \pi\lambda}\right]$$

$$= z_2(Q)$$

$$\frac{\partial^2 K}{\partial z^2} = [IC + \pi\lambda]\sigma\phi(z) > 0$$

Thus, $K(\cdot,Q)$ is convex with a unique minimum at $z_1(Q)$. Hence, if $z < z_1(Q)$, $\frac{\partial K}{\partial z} < 0$, while if $z > z_1(Q)$, $\frac{\partial K}{\partial z} > 0$.

Q.E.D.

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On the convexity of the Hadley-Whitin lost sales inventory model and the characterization of its minimum solution.